

Real hypersurface of a complex space form

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Received: 25 May 2015, Revised: 17 June 2015, Accepted: 17 August 2015

Published online: 6 January 2016

Abstract: The purpose of the present paper is to give characterization of real hypersurface of a complex space form. We find conditions for these hypersurfaces to be ϕ -symmetric and to have η -parallel curvature tensor. Further we prove totally η -umbilical real hypersurfaces of complex space forms have ξ -parallel Ricci tensor and ξ -parallel structure Jacobi operator.

Keywords: Complex space form, real hypersurface, η -parallel, η umbilical, ξ -parallel, structure Jacobi operator.

1 Introduction

A complex $n(\geq 2)$ -dimensional Kählerian manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. The induced almost contact metric structure of a real hypersurface M of $M_n(c)$ is denoted by (ϕ, ξ, η, g) .

In [1] and [2] Berndt has called real hypersurface in $M_n(c)$ with the principal vector ξ as Hopf real hypersurface. It can be easily seen that there does not exist any real hypersurface in $M_n(c)$, $c \neq 0$, which is locally symmetric that is $\nabla R = 0$. This motivates the introduction of the notion of η -parallel curvature tensor (Lee J.G et al [9]). The notion of η -parallel curvature tensor is defined by $g((\nabla_X R)(Y, Z)U, V) = 0$ for any X, Y, Z, U and V in a distribution orthogonal to ξ . This notion is weaker to the notion of η -parallel second fundamental tensor $g((\nabla_X A)Y, Z) = 0$ for any X, Y and Z .

2 preliminaries

Let $M_n(c)$ denote the complex space form of complex dimension n with constant holomorphic sectional curvature $4c$. Let M be a real $(2n-1)$ -dimensional hypersurface immersed in $M_n(c)$ with parallel almost complex structure J and N be unit normal vector field on M . For any vector field X tangent to M , we define ϕ , η and ξ by

$$JX = \phi X + \eta(X)N, JN = -\xi, \quad (1)$$

where ϕX is the tangential part of JX , ϕ is a tensor field of a type $(1, 1)$, η is a 1-form, and ξ is the unit vector field on M . Then they satisfy

$$\begin{aligned} \phi^2 X &= -X + \eta(X)\xi, \phi\xi = 0, \\ \eta(\xi) &= 1, g(X, \xi) = \eta(X), \eta(\phi X) = 0, \\ g(\phi X, \phi Y) &= g(X, Y) - \eta(X)\eta(Y), \\ g(\phi X, Y) &= -g(X, \phi Y), g(\phi X, X) = 0, \end{aligned}$$

$$(\nabla_X \eta)Y = g(\nabla_X \xi, Y). \quad (2)$$

Thus (ϕ, ξ, η, g) is an almost contact metric structure on M . We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M_n(c)$ and by ∇ the one in M determined by the induced metric. Then the Gauss and Weingarten formulae are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad (3)$$

$$\tilde{\nabla}_X N = -AX, \quad (4)$$

for any vector fields X and Y tangent to M . We call A the shape operator of M . For the almost contact metric structure on M , we have

$$\nabla_X \xi = \phi AX, \quad (5)$$

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi. \quad (6)$$

We denote by R the Riemannian curvature tensor field of M . Then the equation of Gauss is given by

$$\begin{aligned} R(X, Y)Z = & \frac{c}{4} [g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ & - 2g(\phi X, Y)\phi Z] + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \quad (7)$$

and the equation of Codazzi is

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4} [\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi].$$

If the shape operator A of M is of the form $AX = aX + b\eta(X)\xi$ for some functions a and b , then M is said to be totally η -umbilical. We put $\alpha = g(A\xi, \xi)$. If ξ is a principal vector everywhere (i.e. $A\xi = \alpha\xi$), we say that M is a Hopf hypersurface and M is called a hypersurface with recurrent shape operator if there exists a 1-form α such that A of M satisfies $(\nabla_X A)Z = \alpha(X)AZ$.

3 Real hypersurfaces of complex space forms with recurrent second fundamental form

Let M be a real hypersurface in a complex space form $M_n(c)$, $c \neq 0$. Taking covariant derivative of (7) with respect to X , we get

$$\begin{aligned} (\nabla_X R)(Y, Z)U = & \frac{c}{4} [\eta(Z)g(AX, U)\phi Y - \eta(U)g(AX, Z)\phi Y + \eta(Y)g(\phi Z, U)AX \\ & - g(AX, Y)g(\phi Z, U)\xi - \eta(Y)g(AX, U)\phi Z + \eta(U)g(AX, Y)\phi Z - \eta(Z)g(\phi Y, U)AX \\ & + g(AX, Z)g(\phi Y, U)\xi - 2\eta(Y)g(AX, Z)\phi U + 2\eta(Z)g(AX, Y)\phi U - 2\eta(U)g(\phi Y, Z)AX \\ & + g(AX, U)g(\phi Y, Z)\xi] + g((\nabla_X A)Z, U)AY + (\nabla_X A)Yg(AZ, U) - g((\nabla_X A)Y, U)AZ \\ & - g((\nabla_X A)Z, U)AZ. \end{aligned} \quad (8)$$

Contracting the above equation with respect to W , we get

$$\begin{aligned}
 g((\nabla_X R)(Y, Z)U, W) &= \frac{c}{4} [\eta(Z)g(AX, U)g(\phi Y, W) - \eta(U)g(AX, Z)g(\phi Y, W) \\
 &+ \eta(Y)g(AX, W)g(\phi Z, U) - \eta(W)g(AX, Y)g(\phi Z, U) - \eta(Y)g(AX, U)g(\phi Z, W) \\
 &+ \eta(U)g(AX, Y)g(\phi Z, W) - \eta(Z)g(AX, W)g(\phi Y, U) + \eta(W)g(AX, Z)g(\phi Y, U) \\
 &- 2\eta(Y)g(AX, Z)g(\phi U, W) + 2\eta(Z)g(AX, Y)g(\phi U, W) - 2\eta(U)g(AX, W)g(\phi Y, Z) \\
 &+ 2\eta(W)g(AX, U)g(\phi Y, Z)] + g((\nabla_X A)Z, U)g(AY, W) + g((\nabla_X A)Y, W)g(AZ, U) \\
 &- g((\nabla_X A)Y, U)g(AZ, W) - g((\nabla_X A)Z, U)g(AZ, W).
 \end{aligned} \tag{9}$$

If the shape operator A is recurrent, then we have

$$\begin{aligned}
 g((\nabla_X R)(Y, Z)U, W) &= \frac{c}{4} [\eta(Z)g(AX, U)g(\phi Y, W) - \eta(U)g(AX, Z)g(\phi Y, W) \\
 &+ \eta(Y)g(AX, W)g(\phi Z, U) - \eta(W)g(AX, Y)g(\phi Z, U) - \eta(Y)g(AX, U)g(\phi Z, W) \\
 &+ \eta(U)g(AX, Y)g(\phi Z, W) - \eta(Z)g(AX, W)g(\phi Y, U) + \eta(W)g(AX, Z)g(\phi Y, U) \\
 &- 2\eta(Y)g(AX, Z)g(\phi U, W) + 2\eta(Z)g(AX, Y)g(\phi U, W) - 2\eta(U)g(AX, W)g(\phi Y, Z) \\
 &+ 2\eta(W)g(AX, U)g(\phi Y, Z)] + 2\alpha(X)[g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)].
 \end{aligned}$$

For all X, Y, Z, U, W orthogonal to ξ , the above equation reduces to

$$g((\nabla_X R)(Y, Z)U, W) = 2\alpha(X)[g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)].$$

Thus we have

Proposition 1. A real hypersurface of a complex space form with recurrent second fundamental form has η -parallel curvature tensor if and only if

$$[g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)] = 0 \text{ holds.}$$

Applying ϕ^2 to (1), we get

$$\begin{aligned}
 \phi^2(\nabla_X R)(Y, Z)U &= \frac{c}{4} [-\eta(Z)g(AX, U)\phi Y + \eta(U)g(AX, Z)\phi Y - \eta(Y)g(\phi Z, U)AX \\
 &+ \eta(Y)\eta(AX)g(\phi Z, U)\xi + \eta(Y)g(AX, U)\phi Z - \eta(U)g(AX, Y)\phi Z + \eta(Z)g(\phi Y, U)AX \\
 &- \eta(Z)\eta(AX)g(\phi Y, U)\xi + 2\eta(Y)g(AX, Z)\phi U - 2\eta(Z)g(AX, Y)\phi U + 2\eta(U)g(\phi Y, Z)AX \\
 &- 2\eta(U)\eta(AX)g(\phi Y, Z)\xi] - g((\nabla_X A)Z, U)AY + \eta(AY)g((\nabla_X A)Z, U)\xi - g(AZ, U)(\nabla_X A)Y \\
 &+ \eta((\nabla_X A)Y)g(AZ, U)\xi + g((\nabla_X A)Y, U)AZ - \eta(AZ)g((\nabla_X A)Y, U)\xi + g(AY, U)(\nabla_X A)Z \\
 &- \eta((\nabla_X A)Z)g(AY, U)\xi.
 \end{aligned} \tag{10}$$

If the shape operator A is recurrent, then (10) takes the form

$$\begin{aligned}
 \phi^2(\nabla_X R)(Y, Z)U &= \frac{c}{4} [-\eta(Z)g(AX, U)\phi Y + \eta(U)g(AX, Z)\phi Y - \eta(Y)g(\phi Z, U)AX \\
 &+ \eta(Y)\eta(AX)g(\phi Z, U)\xi + \eta(Y)g(AX, U)\phi Z - \eta(U)g(AX, Y)\phi Z \\
 &+ \eta(Z)g(\phi Y, U)AX - \eta(Z)\eta(AX)g(\phi Y, U)\xi + 2\eta(Y)g(AX, Z)\phi U \\
 &- 2\eta(Z)g(AX, Y)\phi U + 2\eta(U)g(\phi Y, Z)AX - 2\eta(U)\eta(AX)g(\phi Y, Z)\xi] \\
 &- 2\alpha(X)[g(AY, U)AZ - g(AZ, U)AY].
 \end{aligned} \tag{11}$$

For all X, Y, Z, U, W orthogonal to ξ , the above equation reduces to

$$\phi^2(\nabla_X R)(Y, Z)U = 2\alpha(X)[g(AY, U)AZ - g(AZ, U)AY]. \quad (12)$$

Thus we have

Proposition 2. *A real hypersurface of a complex space form with recurrent second fundamental form is ϕ -symmetric if and only if $[g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)] = 0$ holds.*

The condition ϕ -symmetry on R is weaker to η -parallel condition on R . But these two conditions are equivalent as stated in the following. Combining proposition 1 and proposition 2, we can state the following:

Theorem 1. *In a real hypersurface M of a complex space form $M_n(c)$ with recurrent second fundamental form A satisfying the condition $[g(AZ, U)g(AY, W) - g(AY, U)g(AZ, W)] = 0$, the following are equivalent:*

- (i) M has η -parallel curvature tensor.
- (ii) M is ϕ -symmetric.

4 Totally η -umbilical real hypersurfaces of complex space forms

Let M be a totally η -umbilical hypersurface of the complex space form $M_n(c)$. Then using 10, we get

$$\begin{aligned} g((\nabla_X R)(Y, Z)U, W) = & \frac{c}{4} [\eta(Z)g(AX, U)g(\phi Y, W) - \eta(U)g(AX, Z)g(\phi Y, W) \\ & + \eta(Y)g(AX, W)g(\phi Z, U) - \eta(W)g(AX, Y)g(\phi Z, U) \\ & - \eta(Y)g(AX, U)g(\phi Z, W) + \eta(U)g(AX, Y)g(\phi Z, W) \\ & - \eta(Z)g(AX, W)g(\phi Y, U) + \eta(W)g(AX, Z)g(\phi Y, U) \\ & - 2\eta(Y)g(AX, Z)g(\phi U, W) + 2\eta(Z)g(AX, Y)g(\phi U, W) \\ & - 2\eta(U)g(AX, W)g(\phi Y, Z) + 2\eta(W)g(AX, U)g(\phi Y, Z)] \\ & + (Xa)g(Z, U)g(aY, W) + (Xa)\eta(Y)\eta(W)g(Z, U) \\ & + (Xb)\eta(Z)\eta(U)g(aY, W) + b\eta(U)g(\phi AX, Z)g(aY, W) \\ & + b\eta(Z)g(\phi AX, U)g(aY, W) + (Xa)g(aZ, U)g(Y, W) \\ & + (Xb)\eta(Y)\eta(W)g(aZ, U) + b\eta(W)g(\phi AX, Y)g(aZ, U) \\ & + b\eta(Y)g(\phi AX, W)g(aZ, U) + b(Xa)\eta(Z)\eta(U)g(Y, W) \\ & - (Xa)g(Y, U)g(aZ, W) - (Xb)\eta(Y)\eta(U)g(aZ, W) \\ & - b\eta(U)g(\phi AX, Y)g(aZ, W) - b\eta(Y)g(\phi AX, U)g(aZ, W) \\ & - b\eta(Y)g(\phi AX, U)g(aZ, W) - b(Xa)\eta(Z)\eta(W)g(Z, U) \\ & - (Xa)g(Z, W)g(aY, U) - (Xb)\eta(Z)\eta(W)g(aY, U) \\ & - b\eta(W)g(\phi AX, Z)g(aY, U) - b\eta(Z)g(\phi AX, W)g(aY, U) \\ & - b(Xa)\eta(Y)\eta(U)g(Z, W). \end{aligned} \quad (13)$$

Then for all X, Y, Z, U, W orthogonal to ξ , we have

$g((\nabla_X R)(Y, Z)U, W) = 0$. i.e. M has η -parallel curvature tensor.

Thus we have

Theorem 2. *A totally η -umbilical real hypersurface of a complex space form has η -parallel curvature tensor.*

Now from (1), for a totally η -umbilical real hypersurface, we have

$$\begin{aligned} \phi^2(\nabla_X R)(Y, Z)U = & \frac{c}{4}[-\eta(Z)g(AX, U)\phi Y + \eta(U)g(AX, Z)\phi Y - \eta(Y)g(\phi Z, U)AX \\ & + \eta(Y)\eta(AX)g(\phi Z, U)\xi + \eta(Y)g(AX, U)\phi Z - \eta(U)g(AX, Y)\phi Z + \eta(Z)g(\phi Y, U)AX \\ & - \eta(Z)\eta(AX)g(\phi Y, U)\xi + 2\eta(Y)g(AX, Z)\phi U - 2\eta(Z)g(AX, Y)\phi U + 2\eta(U)g(\phi Y, Z)AX \\ & - 2\eta(U)\eta(AX)g(\phi Y, Z)\xi] - (Xa)g(Z, U)aY - (Xb)\eta(Z)\eta(U)aY - b\eta(U)g(\phi AX, Z)aY \\ & - b(Xa)\eta(Y)g(Z, U)\xi + \eta(aY)(Xa)g(Z, U)\xi + (Xb)\eta(Z)\eta(U)\eta(aY)\xi - g(aZ, U)(Xa)Y \\ & - b\eta(Y)g(aZ, U)\phi AX - b\eta(Z)\eta(U)(Xa)Y + (Xa)\eta(Y)g(aZ, U)\xi + (Xa)g(Y, U)aZ \\ & + (Xb)\eta(U)\eta(Y)aZ + b\eta(U)g(\phi AX, Y)aZ + \eta(Y)g(\phi AX, U)aZ - b(Xa)\eta(aZ)g(Y, U)\xi \\ & - (Xb)\eta(aZ)\eta(Y)\eta(U)\xi - b\eta(U)\eta(aZ)g(\phi AX, Y)\xi - b\eta(Y)\eta(aZ)g(\phi AX, U)\xi \\ & + g(aY, U)(Xa)Z + \eta(Z)g(aY, U)\phi AX + b\eta(Y)\eta(U)(Xa)Z - \eta(Z)g(aY, U)(Xa)\xi. \end{aligned}$$

For all X, Y, Z, U, W orthogonal to ξ , the above equation reduces to

$$\phi^2(\nabla_X R)(Y, Z)U = -(Xa)g(Z, U)aY - g(aZ, U)(Xa)Y + (Xa)g(Y, U)aZ + g(aY, U)(Xa)Z.$$

It is well known that:

Remark 1. Any totally η -umbilical real hypersurface is a Hopf hypersurface.

Remark 2. In a Hopf hypersurface with ξ as a principal vector, principal curvature corresponding to ξ is a constant.

From the above remarks we have for constant a ,

$$\phi^2(\nabla_X R)(Y, Z)U = 0.$$

Thus we can state that

Theorem 3. A totally η -umbilical real hypersurface of a complex space form is ϕ -symmetric.

Now let us denote by Q the Ricci tensor of M in $M_n(c)$. Then from (7) which together with (5), we obtain

$$QX = \frac{c}{4}[(2n+1)X - 3\eta(x)\xi] + hAX - A^2X,$$

$$(\nabla_X Q)Y = -\frac{3c}{4}[g(\phi AX, Y)\xi + \eta(Y)\phi AX] + (Xh)AY + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY, \quad (14)$$

where I denote the identity map on the tangent space $T_p M$, $p \in M$.

If M is totally η -umbilical, then from (4) we obtain

$$(\nabla_\xi Q)Y = (\xi h)c\xi + (hI - A)((\xi a)Y + (\xi b)\eta(Y)\xi) - ((\xi a)AY + (\xi b)\eta(AY)\xi),$$

By remark 1 and remark 2, we have $(\nabla_\xi Q)Y = 0$, provided $a = -b$.

Theorem 4. A totally η -umbilical real hypersurface of a complex space form has ξ -parallel Ricci tensor provided $a = -b$.

From (5), we obtain

$$R_{\xi}Y = R(Y, \xi)\xi = \frac{c}{4}[Y - \eta(Y)\xi] + \alpha AY - \eta(AY)A\xi \quad (15)$$

for any vector field Y on M . Taking covariant derivative with respect to X of (15), we get

$$\begin{aligned} (\nabla_X R_{\xi})Y &= -\frac{c}{4}[(\nabla_X \eta)Y] - \frac{c}{4}[\eta(Y)(\nabla_X \xi + (X\alpha)AY \\ &\quad + \alpha((\nabla_X A)Y) - ((\nabla_X \eta)AY)A\xi - (\eta(\nabla_X A)Y)A\xi \\ &\quad - \eta(AY)\nabla_X A\xi - \eta(AY)A(\nabla_X \xi)] \end{aligned}$$

Contracting the above with respect to Z and using (4) and (5), we get

$$\begin{aligned} g((\nabla_X R_{\xi})Y, Z) &= -\frac{c}{4}[g(\nabla_X \xi, Y)\eta(Z) + \eta(Y)g(\nabla_X \xi, Z)] \\ &\quad + X\alpha g(AY, Z) + \alpha g((\nabla_X A)Y, Z) + g(X, A\phi AY)g(A\xi, Z) \\ &\quad - g((\nabla_X A)Y, \xi)g(A\xi, Z) - g(AY, \xi)g((\nabla_X A)\xi, Z) + g(AY, \xi)g(X, A\phi AZ). \end{aligned}$$

Since M is totally η -umbilical, the above equation reduces to

$$\begin{aligned} g((\nabla_X R_{\xi})Y, Z) &= -\frac{c}{4}[g(\phi AX, Y)\eta(Z) + \eta(Y)g(\phi AX, Z)] + (X\alpha)[ag(AY, Z) \\ &\quad + b\eta(Y)\eta(Z)]\alpha[(Xa)g(Y, Z) + (Xb)\eta(Y)\eta(Z) + b(g(\phi AX, Y)\eta(Z) \\ &\quad + \eta(Y)g(\phi AX, Z))] + \eta(AZ)g(X, A\phi AY) - [(Xa)\eta(Y) \\ &\quad + (Xb)\eta(Y) + bg(\phi AX, Y)]\eta(AZ) - [(Xa)\eta(Z) \\ &\quad + (Xb)\eta(Z) + bg(\phi AX, Z)]\eta(AY) + \eta(AY)g(X, A\phi AZ). \end{aligned}$$

For all X, Y, Z orthogonal to ξ , we obtain

$$g((\nabla_X R_{\xi})Y, Z) = (X\alpha)ag(AY, Z) + \alpha(Xa)g(Y, Z).$$

If a and b are constants, then we have

$$g((\nabla_X R_{\xi})Y, Z) = 0.$$

Theorem 5. *A totally η -umbilical real hypersurface of a complex space form has ξ -parallel structure Jacobi operator provided the associated scalars a and b are constants.*

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